

Characterizing Optimal Syndicated Sponsored Search Market Design

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Abstract

The analysis to date of sponsored search auctions, originally conducted by Varian and independently by Aggarwal et al., has yet to describe more recent developments such as the emergence of interacting advertising exchanges. This paper assesses the significance of such phenomena. Building on Gonen and Vassilvitskii's model of sponsored search with reserve prices we depict advertising networks as double-sided sponsored search markets with advertisers on one side, syndicators on the other, and the search engine as the market maker. We call this market the *syndicated sponsored search market*.

We focus our attention on investigating the impact of the common assumption of separability on the optimal syndicated sponsored search market. Though the optimal sponsored search market under the separability assumption follows from a Myerson-like mechanism, when the separability assumption is removed an impossibility is revealed. We present a full characterization of the truthful syndicated sponsored search market, showing VCG-like prices in the optimal market and indicating that no truthful budget-balance/surplus syndicated sponsored search market can be designed if separability does not exist.

Characterizing truthful syndicated sponsored search markets requires the use of a relatively new set of tools, reductions that preserve economic properties. This paper utilizes two such reductions; a *truth-preserving reduction* and a *non-affine preserving reduction*. The truth-preserving reduction is used to reduce the syndicated sponsored search market to a special case of a sub-additive combinatorial auction to allow us to make use of the impossibility result proved in [5]. Intuitively, our proof shows that truthful syndicated sponsored search markets, where separability is not assumed, are as hard to design as truthful subadditive combinatorial auctions with multi-minded payers.

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1 Introduction

This paper investigates the impact of the common assumption of separability in the optimal syndicated sponsored search market. In such markets the buyers are advertisers and the sellers are syndicators offering advertising slots. Though the optimal sponsored search market under the separability assumption follows from a Myerson-like mechanism, removing the separability assumption reveals an impossibility. A full characterization of the truthful sponsored search syndicated market is given showing VCG-like prices in the optimal market and indicating that no truthful budget balance/surplus syndicated sponsored search market can be designed if separability does not exist.

Sponsored search auctions are the primary way that companies like Google, Microsoft and Yahoo monetize their search engines. Such auctions allow advertisers to bid on particular queries, thereby ensuring the relevance of the advertisement to the user, and increasing the conversion rate. Sponsored search has been a large and constantly growing business in recent years, and is expected to see continued growth. Not surprisingly the scale of sponsored search business has led to much research interest in the analysis of the precise way the auctions are run, (i.e. [1, 10, 3, 6, 14, 12, 7, 11]).

Sponsored search auctions are based on a very simple framework¹. Each advertiser specifies the query she is looking to advertise on, and submits a bid, representing the maximum amount she is willing to pay. When a user enters a query, the system collects all of the advertisers bidding for the query, and runs a generalized second-price auction to determine both the winners, and the prices that each would be charged. There are usually multiple winners, as there are multiple advertising slots on each search results page, with higher slots being more valuable since they are seen by more users. Finally, the advertiser is charged only in the event of a user click on the advertiser's ad, otherwise

¹In practice the situation is much more complex - e.g. advertisers specify maximum daily budgets, there is fuzzy matching on the queries, etc. We do not consider these problems in this work.

no money changes hands, the so-called pay per click scheme.

Separability - One of the first analyses of these auctions, [3] showed that the greedy ranking employed by Google agrees with the efficient allocation only when click-through rates are separable, that is, click through rates are the product of the function of the advertiser quality and the position in which the advertisement appeared. The separability property has since been used as a simplifying assumption in most other work on sponsored search i.e., [6, 18, 7, 10, 11] and many more. Recently some work has investigated the limitation of the separability assumption and suggested an extended separability assumption to allow for richer models of truthful sponsored search auctions with reserve prices for the slots [12]. Other work gave alternative models that do not follow the separability assumption and better describe the auctions used in practice [4, 2].

Syndicated Market - As the sponsored search market evolves into networks of advertisers and syndicators a natural question arises: is it possible to conduct a market where all parties, i.e., the buyers and the seller are simultaneously motivated to tell the truth. Following Myerson-Satterthwaite’s result [16], the market maker will potentially sustain a budget deficit. As the market maker in the desired syndicated market is also the search engine it is unreasonable to expect the market maker to carry a loss. One way to overcome Myerson-Satterthwaite’s impossibility is to give up some efficiency and maintain the other properties of individual rationality (no player losses by participating in the market), truthfulness and budget balance/surplus.

Interestingly the question of maintaining budget balance in a truthful syndicated sponsored search market ties back to the separability condition. While under separability it is possible to create a truthful budget balanced/surplus syndicated sponsored search market as was shown in [12], the question of designing such market where separability is not assumed was left open. [12]’s truthful, individually rational, and budget balanced/surplus syndicated sponsored search market gives up some market efficiency and assumes separability. The separability assumption enables a simplification to the problem domain and allows players to describe their private values by a single number. The set of problems where players can describe their values by a single number are called single-parameter domains and were studied in number of algorithmic game theory contexts, such as combinatorial auctions. When separability is not assumed the problem domain can not be

simplified and therefore the problem of the syndicated sponsored search problem is a multi-parameter domain.

Characterization of optimal multi-parameter domains

For single-parameter domains, Myerson’s characterization [15] tells us that a social choice function, the market mechanism rule for determining the outcome, is truthful if and only if it is monotone; for a monotone social choice function a player’s winnings can only increase when his bid increases (for a fixed set of the other players’ bids). For truthful single-parameter mechanisms Myerson provides a simple way to compute the revenue maximizing truthful mechanism, i.e., the optimal mechanism. Unfortunately, much less is understood about exactly which social choice functions are truthful in multi-parameter domains.

The main characterization result in such domains is Roberts’ Theorem [17]. Roughly, it says that if the v_i ’s (the players’ valuation of the outcome) can be arbitrary with no structure (unrestricted valuations), only a very small subset of functions are truthful. These functions are called affine maximizers. An affine maximizer has the following form: $\arg \max_{a \in A} \omega_i v_i(a) + \gamma_a$. Where A is the set of possible outcomes. Notice that the γ_a ’s and the ω_i ’s are predetermined constants and do not depend on the players’ valuations. Interestingly, this is precisely the set of functions that can be made truthful by attaching a (weighted) VCG pricing scheme. Unfortunately, unrestricted valuations are not applicable in most settings (for instance in many auctions bidders have no externalities and only care about the items they receive) and Roberts’ theorem does not apply. Some partial success was made in characterizing multi-parameter restricted domains; for combinatorial auctions [13], for subadditive combinatorial auctions [5] and for matching markets [9]. This paper considers a new multi-parameter restricted domain in the context of sponsored search - the syndicated sponsored search market.

1.1 Our Results

We focus our attention to the problem of *Sponsored Search Syndicated Markets*. In such markets the n buyers are advertisers and the k sellers are syndicators offering advertising slots. Every buyer b_i has a privately known value $v_i : A \rightarrow \mathbb{R}^+$ for a click on having his ad allocated to one of the slots and has a private estimate on the click-through rate $\lambda_{i,j} : A \rightarrow [0, 1]$ of his ad in

the given slot j^2 . Every syndicator s_j has a private cost $c_j : A \rightarrow \mathbb{R}^-$ for selling a slot.

We investigate the impact of the common assumption of separability in the optimal market, i.e., a revenue-maximizing truthful sponsored search syndicated market. We show that while assuming separability the revenue maximizing market is relatively straightforward to compute from Myerson’s theorem [15], and when separability is *not* assumed it is no longer possible to use Myerson to compute a revenue maximizing solution. Without the separability assumption VCG-like mechanisms produce the maximum revenue that can be achieved for a truthful sponsored search syndicated market. This is concluded by our main impossibility result characterizing the truthful sponsored search syndicated markets for buyers; showing essentially that every such truthful market is an affine maximizer for buyers i.e., a truthful market will result in VCG-like prices for buyers.

Theorem: *The social choice function of any truthful syndicated sponsored search market mechanism for buyers that does not assume separability is an affine maximizer.*

Our work builds on Roberts’ results and the results achieved by [5] integrating reductions, a classic tool of computer science theory. Although reductions are widely used in proving the hardness of problems by reducing them to other well characterized hard problems, to date there is only one other example of the use of reductions in mechanism design [9]³. This work makes use of the negative result in [5] for subadditive combinatorial auctions to show the same negative result for syndicated sponsored search markets by the means of a reduction.

The task of building a reduction between subadditive combinatorial auctions and syndicated sponsored search markets is somewhat involved; a general diagram of the proof can be found at 1. Since our theorem shows that the syndicated sponsored search market mechanism with the property of truthfulness implies affine maximization, we need to construct a reduction that maintains the truthfulness property and the non affine property. We use the reduction concept of a *truth-*

²The motivation behind the advertisers having a privately known estimate on the click-through rate is that each advertiser has private knowledge of which ad he will place in the slot and possibly also has some knowledge of his competitors’ ad offering from previous auctions or markets.

³Except for being implicitly used by the revelation principle’s proof.

preserving reduction and a *non-affine preserving reduction* defined by [9] in the context of mechanism design.

Our proof that any truthful syndicated sponsored search market’s (S3M) social choice function that does not assume separability is affine maximizing for buyers is composed of two reductions and makes use of two additional problems. The first step of the proof utilizes the theorem of [5] for a special case of subadditive combinatorial auction which we call the *combinatorial auction subadditive (CASA)*.

Lemma: *The social choice function of any truthful CASA mechanism is an affine-maximizer.*

We then utilize a reduction from a special case of combinatorial matching market composed of sellers with cost zero for their goods, which we call *Combinatorial Matching Cost 0 problem (CMC0)* to CASA. This reduction preserves the truthfulness and the non-affine properties. This reduction will then yield the following theorem:

Lemma: *The social choice function of any truthful CMC0 mechanism is an affine maximizer.*

Once any truthful CMC0 is shown to be affine maximizing a reduction from S3M to CMC0 is constructed which again preserves the truthfulness and non-affine properties. As the sellers’ values (and therefore the sellers’ prices) in the CMC0 are zero, the reduction $S3M \leq CMC0$ preserves the non-max affine property only for the S3M buyers.

We extend the characterization of the truthful sponsored search syndicator market to include sellers. To prove the max-affine maximizing property for the sellers as well, we define and perform a *price expansion* of the induced buyers’ prices and define critical value prices for sellers. The critical prices are shown to be truthful affine-maximizing prices. The model of S3M assumes unique ownership of sellers over slots, a syndicator model (MSS) with seller who can sell any slot is presented. By showing that the MSS is reducible to S3M, and that the reduction is truthful non-affine maximizing preserving, it follows that:

Lemma: *The social choice function of any truthful MSS mechanism for sellers is an affine maximizer.*

To complete the structure we prove that the integration of two price vectors that are truthful and affine maximizing is also truthful and affine maximizing. This then yields our result and answers the open question at [12], that there can not exist a budget balanced/surplus

sponsored search syndicated market without the separability assumption.

Theorem: *The social choice function of any truthful syndicated sponsored search market mechanism that does not assume separability is an affine maximizer.*

Organization: The body of the paper is organized as follows. The next section gives notations, and defines the concepts of a truth-preserving reduction and non-affine preserving reduction. Section 3 defines the CASA, CMC0 and S3M problems. Section 4 gives the road map for the three parts of the paper; the separable case, the non-separable case for buyers and the non-separable case for sellers. Section 5 presents the optimal market for the separable case, while subsections 6.1 6.2 and 6.3 of section 6 gives the characterization of the CASA, CMC0 and S3M for buyers problems respectively. Section 7 gives the characterization of the S3M problem for sellers. Finally, in section 8 we prove the integrated buyers, sellers non-separable case and answer the open question in [12].

Due to lack of space the technical details of some of the proofs are deferred to the working paper [8].

2 Setting and Notations

In this paper we characterize the properties of the syndicated sponsored search market (S3M). In the S3M problem there are k slots of advertisement. There is a cost associated with each slot and each slot might be sold by a different seller. There are n buyers potentially interested to be allocated only to a single slot but are interested to consider any of the locations. Each buyer has a value for getting a click on his ad regardless of its location. Each buyer also associates each of the slots with a probability for his ad to be clicked, i.e., click-through-rate based on his ad and the location. We are interested in a setting where both the value and the click-through-rates of the buyer are his private values and the sellers hold a private cost. We brake our general setting into two cases; one assuming the click-through-rates are separable (i.e., each slot's click-through-rate is the product of the buyer's ad and the slot location) and the other assumes they are not.

To characterize the properties of the S3M problem in the non-separable case we characterize a different problem: the combinatorial auction subadditive (CASA). The CASA has m players and n different goods where each player is interested in subsets of the goods.

We assume that all of our mechanisms have a finite

set of possible outcomes (range of alternatives to choose from) and denote this set by A , where $|A| = l$. Each player i , $1 \leq i \leq n$, assigns a real value $v_i(a)$ to each possible alternative a from A . Namely, $v_i(a)$ is the valuation of player i on an output a . The vector $v_i \in \mathbb{R}^l$ specifies i 's preferences on all possible $a \in A$. The set $V_i \subseteq \mathbb{R}^l$ is the set of all possible valuations v_i on all possible $a \in A$ we refer to V_i also as i 's domain. The set of all possible valuations of all the players is denoted by $V = V_1 \times \dots \times V_n$. Let $v(a) = (v_1(a), \dots, v_n(a)) \in \mathbb{R}^n$ be the vector of valuations of all the players on outcome a . Let $v_{-i} = (v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_n)$ be the vector of valuations of all the players besides player i , and let V_{-i} be the set of all possible vectors v_{-i} .

In this paper we assume players have quasi linear utility namely that player i 's utility is $v_i(a) - p_i(v)$ where $p_i(v)$ is the price player i is charged by the mechanism when alternative a is chosen given v as the valuation vector.

As our main theorem extends the main result in [5] we present our allocations in terms of social choice functions as in [5]. The necessary definitions from [5] used in our paper follows:

DEFINITION 2.1. *Social choice function: Social choice function $\phi : V_1 \times \dots \times V_n \Rightarrow A$ is a function that gets as input a vector of players' preferences and chooses an alternative among a finite set of possible alternatives A . We assume w.l.o.g that ϕ is onto A .*

DEFINITION 2.2. ([13]) *Truthfulness: A mechanism (ϕ, p_1, \dots, p_n) , where $\phi : V \rightarrow A$ and $p_z : V \rightarrow \mathbb{R}$ is called truthful if for any player z , any $v_{-z} \in V_{-z}$, and any $v_z, u_z \in V_z$ it holds that $v_z(\phi(v)) - p_z(v) \geq v_z(\phi(u_z, v_{-z})) - p_z(u_z, v_{-z})$. The social choice function ϕ is implementable or simply truthful if there exist some mechanism that implements it.*

DEFINITION 2.3. ([13]) *Affine maximization: A social choice function ϕ is an affine maximizer if there exist constants $\omega_1, \dots, \omega_n \geq 0$ and $\{\gamma_a\}_{a \in A}$ such that for any $v \in V : \phi(v) \in \arg \max_{a \in A} \{\sum_{z=1}^n \omega_z v_z(a) + \gamma_a\}$. In this case ϕ is implemented by the prices $p_z = -\omega_z^{-1}(\sum_{k \neq z} \omega_k v_k(a) + \gamma_a)$.*

DEFINITION 2.4. *Critical price: A mechanism uses a critical price payment scheme if given an allocation it charges players the minimum value they need to report to the mechanism in order to receive the same allocation.*

DEFINITION 2.5. ([5]) *Large Range: A mechanism is said to have large enough range if the social choice func-*

tion must select an outcome from at least $n+2$ different outcomes.

DEFINITION 2.6. ([5]) *Stable Mechanism:* Let f be a mechanism for combinatorial auctions domain. The mechanism f is stable if for each bidder i , and each two of his possible valuations v, v' the following holds: let $f(v, v_i) = (S_1, \dots, S_n)$ and $f(v', v_i) = (S'_1, \dots, S'_n)$. If $S_i = S'_i$, then $S_k = S'_k$, for all $k \neq i$.

DEFINITION 2.7. ([5]) *Scalable Mechanism:* A social choice function satisfies scalability if the outcome does not change when all valuations are scaled uniformly.

In order to prove our main theorem regarding the non-separable setting of S3M we reduce the S3M mechanism to (a variation of) a subadditive multi-minded combinatorial auction. Moreover, we prove that the reduction maintains the necessary properties (in order to use [5]'s theorem) of the combinatorial auction, i.e., truthfulness and non-affine maximization. To prove that our reduction preserves the desired properties, we use the concepts of reduction previously defined in [9]. For completeness the definitions are stated below:

DEFINITION 2.8. ([9]) *A social choice function ϕ is reducible to a choice function $\bar{\phi}$, namely, $\phi \leq \bar{\phi}$, if ϕ 's input can be reduced to $\bar{\phi}$'s input such that the target function of ϕ is optimum if and only if the target function of $\bar{\phi}$ on the reduced input is optimum.*

DEFINITION 2.9. *Truth Preserving Reduction:* ([9]) Given mechanisms $\alpha = (\phi, p_1, \dots, p_n)$ and $\beta = (\bar{\phi}, \bar{p}_1, \dots, \bar{p}_m)$, a reduction $\alpha \leq \beta$ is a truth preserving reduction if there exist a function $h : \phi \rightarrow \bar{\phi}$ such that $\phi \leq \bar{\phi}$ and for every $1 \leq i \leq m$ there exists a function $g_i : \{p_1, \dots, p_n\} \rightarrow \bar{p}_i$ s.t. if (ϕ, p_1, \dots, p_n) is truthful then $(h(\phi), g_1(p_1 \dots p_n), \dots, g_m(p_1 \dots p_n))$ is truthful.

DEFINITION 2.10. *Non-affine maximizing preserving reduction:* ([9]) Given social choice functions ϕ and $\bar{\phi}$, a reduction $\phi \leq \bar{\phi}$ is a non-affine maximizing preserving reduction if the following holds: if ϕ is a non-affine maximizing social choice function then $\bar{\phi}$ is a non-affine maximizing social choice function.

3 Problem Definition

In this section we give the formal definitions of the problems we use in our reductions.

Problem 1. CASA:

Let \tilde{D} be a set of m players, and let \tilde{G} be a set of goods

of size n . Namely, $\tilde{D} = \{\tilde{d}_z | 1 \leq z \leq m\}$ and $\tilde{G} = \{\tilde{g}_i | 1 \leq i \leq n\}$. For every player $\tilde{d}_z \in \tilde{D}$ there is a vector of mappings

$V_z = (\tilde{v}_z^{\{1\}}, \dots, \tilde{v}_z^{\{n\}}, \tilde{v}_z^{\{1,1\}}, \dots, \tilde{v}_z^{\{n,n\}}, \dots, \tilde{v}_z^{\tilde{G}})$ s.t. $\tilde{v}_z^{\{Q\}} : A \rightarrow \mathbb{R}^+$ such that \tilde{d}_z is allocated a bundle of goods in allocation $a \in A$ and the following hold: for all sets $Q \in 2^{\tilde{G}}$, $|Q| = 1$ such that \tilde{d}_z is allocated the bundle Q in allocation $a \in A$ then $\tilde{v}_z^{\{i\}}(a) \geq 0$, if $|Q| > 1$ such that \tilde{d}_z is allocated the bundle Q in allocation $a \in A$ then $\tilde{v}_z^{\{Q\}}(a) \leq \sum_{\tilde{g}_i \in Q} \tilde{v}_z^{\{\tilde{g}_i\}}(a)$.

The n goods are allocated to the m players, namely, player $\tilde{d}_z \in \tilde{D}$ receives a set of goods $Q \in 2^{\tilde{G}}$ in allocation $a \in A$ s.t. $\sum_{z, Q \in 2^{\tilde{G}}} \tilde{v}_z^{\{Q\}}(a) \cdot x_z^Q$ is maximum, under the following constraints:

1. for all $1 \leq z \leq m$, and $Q \in 2^{\tilde{G}}$, $x_z^Q \in \{0, 1\}$.
2. for all $1 \leq z \leq m$, $\sum_{Q \in 2^{\tilde{G}}} x_z^Q \leq 1$
3. for all $1 \leq i \leq n$, $\sum_{z, \tilde{g}_i \in Q} x_z^Q \leq 1$.

The meaning of the constraints is that any good can not be allocated more than once (constrain 3). A player can be allocated only one subset of goods (constraint 2).

Problem 2. S3M: (S3M -Syndicated Sponsored Search Market)

Let $S = \{s_1, \dots, s_k\}$ be a set of sellers each having a single good. Let $G = \{g_1, \dots, g_k\}$ be the set of goods, where g_j denotes the good of seller s_j . For each seller s_j $1 \leq j \leq k$ there is mapping $c_j : a \rightarrow \mathbb{R}^-$ such that g_j is allocated in allocation $a \in A$. Let $B = \{b_1, \dots, b_n\}$ be a set of buyers. For each $1 \leq i \leq n$ there is mapping $v_i : a \rightarrow \mathbb{R}^+$ such that b_i is allocated in allocation $a \in A$ and a mapping $\lambda_{i,j} \rightarrow [0, 1]$ for all $1 \leq j \leq k$. Let M be a set of pairs (b_i, s_j) , where $b_i \in B$ and $s_j \in S$, such that M is a matching between B and S .

We want to match buyers to sellers such that the gain from trade (social welfare) of the allocation is maximized, i.e., our target function is

$$\max \sum_{i,j | (b_i, s_j) \in M} \lambda_{i,j} (v_i(a) + c_j(a)).$$

where a is the allocation immersed from M .

Problem 3. CMC0: (CMC0-Combinatorial Matching Cost 0) Let $S' = \{s'_1, \dots, s'_k\}$ be a set of sellers each having a single good. Let $G' = \{g'_1, \dots, g'_k\}$ be the set of goods, where g'_j denotes the good of seller s'_j . For each seller s'_j $1 \leq j \leq k$ there is mapping $c'_j : A \rightarrow 0$ such that g'_j is allocated in allocation $a \in A$. Let $B' =$

$\{b'_1, \dots, b'_n\}$ be a set of buyers. For each $1 \leq i \leq n$ there is mapping $v'_i : a \rightarrow \mathbb{R}^+$ such that b'_i is allocated a pair of goods in $G' \times G'$ at allocation $a \in A$. Meaning that every buyer has a value for every pair of goods (bundle of size 2). Let M be a set of triples (b'_i, s'_j, s'_t) , where $b'_i \in B'$ and $s'_j, s'_t \in S'$, such that M is a matching between B' and S' .

We want to match buyers to sellers such that the gain from trade (social welfare) of the allocation is maximized, i.e., our target function is

$$\max \sum_{i,j,t|(b'_i, s'_j, s'_t) \in M} v'_i(a) + c'_j(a) + c'_t(a).$$

where a is the allocation immersed from M .

We will refer to the prices charged by the S3M mechanism to players in set B as buyers' prices and the prices charged by the S3M mechanism to players in set S as sellers' prices.

The intuition behind our reduction construction of the non-separable setting is that each of the CASA' goods corresponds to a buyer in S3M and each bundle a player is interested in CASA corresponds to a potential match to a certain seller/slot in S3M. By maximizing the welfare in CASA we maximize the gain from trade in the original S3M.

4 Road Map

4.1 The Separable Case

Closely following Myerson's theorem, we show that under the separability assumption we can define virtual values that allow for revenue maximizing truthful syndicated sponsored search market.

4.2 The Non-Separable Case for Buyers

In order to prove our main theorem of the non-separable setting we prove that the social choice function of any truthful CASA mechanism is an affine maximizer. By showing that the CMC0 is reducible to CASA, and that the reduction is truthful non-affine maximizing preserving, it follows that the social choice function of any truthful CMC0 mechanism is affine maximizer. We move on to prove that the non-separable S3M is reducible to CMC0, and that the reduction is truthful non-affine maximizing preserving for buyers. It follows that the social choice function of any truthful non-separable S3M mechanism is an affine maximizer for buyers.

4.3 The Non-Separable Case for Sellers

In order to extend the characterization of the truthful sponsored search syndicator market to include sellers, we prove that the social choice function of any truthful S3M mechanism for sellers is an affine maximizer. The above is proved by defining critical value prices and showing that when applying critical price payment scheme to the sellers in S3M a truthful mechanism that is an affine maximizer for sellers is constructed. The model of S3M assumes unique ownership of sellers over slots, a syndicator model (MSS) with seller who can sell any slot is presented. By showing that the MSS is reducible to S3M, and that the reduction is truthful non-affine maximizing preserving, it follows that the social choice function of any truthful MSS mechanism is affine maximizer. Thus showing that every truthful syndicated sponsored search market that does not assume separability is an affine maximizing.

5 Sponsored Search Syndicated Market: the Separable Case

In this section we show how the revenue maximizing truthful sponsored search syndicated market (S3M) can be computed using Myerson's theorem [15] for optimal auctions while assuming separability. The full details of section 5 can be found in the working paper [8].

6 Sponsored Search Syndicated Market: The Non-Separable Case for Buyers

6.1 Combinatorial Auction SubAdditive (CASA) Characterization

In this section we prove that the social choice function of any truthful CASA mechanism is an affine-maximizer, using the result of [5]:

[5]'s Theorem 5.2: *Every stable and scalable mechanism with a large enough range for combinatorial auctions where bidders have subadditive valuations is an affine maximizer.*

In the working paper [8] we prove that the CASA mechanism maintains the properties required by [5].

LEMMA 6.1. *The social choice function of any truthful CASA mechanism is an affine-maximizer.*

Proof. It follows from lemmas ??, ??, ??, and ?? that CASA satisfy all the necessary and sufficient conditions of theorem 5.2 in [5]. \square

6.2 CMC0 Characterization

In this section we prove that the social choice function of any CMC0 truthful mechanism is an affine-maximizer by showing that CMC0 is reducible to CASA, using a truthful and non-affine-maximizer preserving reduction.

The intuition behind the $\text{CMC0} \leq \text{CASA}$ reduction construction is to turn each buyer in CMC0 to a good in the CASA, and to turn each buyer's value in CMC0 for a match with two sellers to a player's value in CASA for that good (the buyer in CMC0). Each player in CASA values each good following a different pair of sellers valued by the associated buyer in CMC0 with the good in CASA. Thus a bundle allocated to a player in CASA represents a sub matching in CMC0. If the bundle of all goods is allocated to a player in CASA it represents the full matching in CMC0.

Due to lack of space the technical details of the reduction construction, reduction proofs and the properties preserving proofs are deferred to the working paper [8].

6.3 S3M Characterization for Buyers

In this section we prove that the social choice function of any S3M truthful mechanism is an affine-maximizer for buyers by showing that S3M is reducible to CMC0, using a truthful and non-affine-maximizer preserving reduction.

6.3.1 Reduction: $\text{S3M} \leq \text{CMC0}$

In this subsection we prove that S3M is reducible to CMC0.

The intuition behind the $\text{S3M} \leq \text{CMC0}$ reduction construction is to add nk sellers of goods g'_{k+i_j} in addition to turning every seller s_j with valuation greater than 0 for his good g_j in the S3M to n buyers with valuation $-\lambda_{i,j}c_j$, $1 \leq i \leq n$ for the bundle of goods g'_{i_j} and $g'_{j(i)}$, and to n sellers who has the good $g'_{j(i)}$, $1 \leq i \leq n$ but who has a valuation of 0 for the good $g'_{j(i)}$ in the CMC0.

Construction 1. Let S^+ be the set of sellers in the S3M with $-c_j(a) > 0$ where g_j allocated in a . Let S^{++} be the set of additional nk sellers of goods $g'_{k+(i-n)j}$.

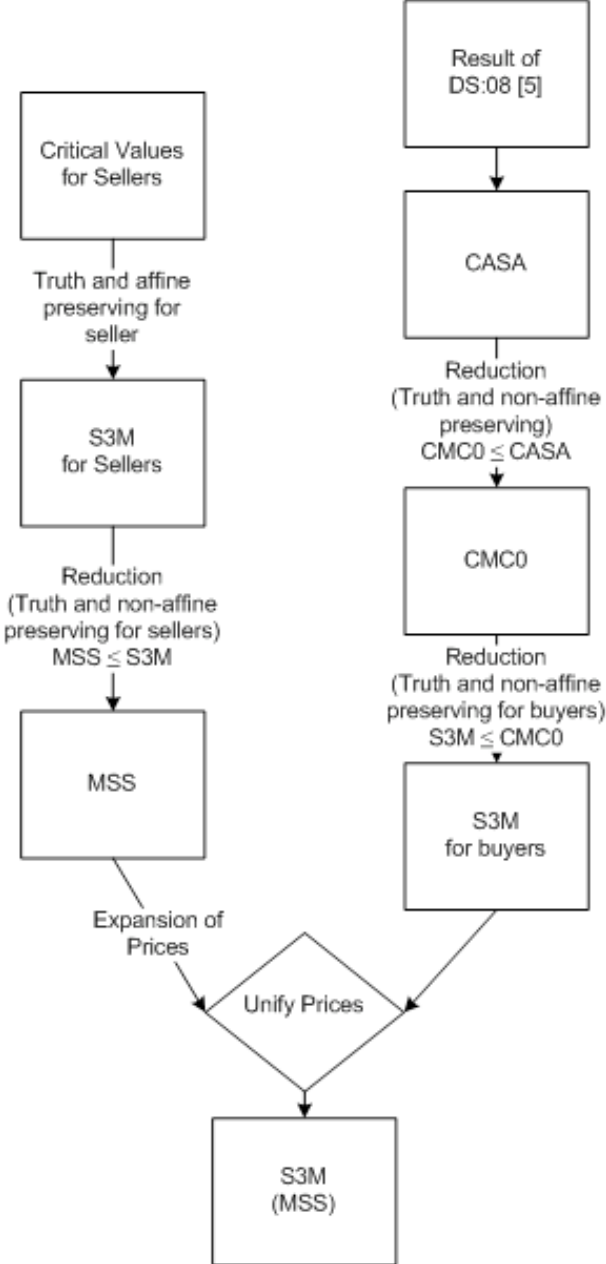


Figure 1: Roadmap of the non-separable proof

We use "–" notation for the reduced input parameters. Then $\bar{S}' = S \cup \{s'_{k+1}, \dots, s'_{k+n_k}, \dots, s'_{|S^+|n+k+n_k}\}$, $\bar{G}' = G \cup \{g'_{k+1}, \dots, g'_{k+n_k}, \dots, g'_{|S^+|n+k+n_k}\}$, and $\bar{B}' = B \cup \{b'_{n+1}, \dots, b'_{|S^+|n+n+1}\}$. In addition, for every $1 \leq j \leq k$, for all $1 \leq i \leq n$ $\bar{v}'_i(\bar{a}) = \lambda_{i,j} v_i(a)$ where b'_i is allocated bundle $\{g'_j, g'_{k+(i-n_j)j}\}$ in \bar{a} and b'_i is allocated g_j in a , and for all $n+1 \leq i \leq |S^+|n+n+1$, $\bar{v}'_i(\bar{a}) = -\lambda_{(i-n_j),j} c_j(a)$ where b'_i is allocated bundle $\{g'_{(i-n_j)j}, g'_{j(i-n_j)}\}$ in \bar{a} , j index of seller in S^+ .

DEFINITION 6.1. Induced Matching: Let M be a matching of the input of the S3M problem. Then the induced matching M_0 of the CMC0 problem is the following matching:

$$(b'_i, s'_j, s'_{k+i_j}) \in M_0 \iff (b_i, s_j) \in M \text{ or } (i \geq n+1 \text{ and } (b'_i, s'_{j(i-n_j)}, s'_{k+(i-n_j)j}) \in M_0 \text{ and } (b'_{i-n_j}, s'_j) \in M \text{ where } j \in S^+).$$

Let M_0 be a matching of the reduced input of the CMC0 problem. Then the induced matching M of the S3M problem is defined in the following manner:

$$(b_i, s_j) \in M \iff (b'_i, s'_j, s'_{k+i_j}) \in M_0 \text{ and } i \leq n.$$

Due to lack of space the technical details of the reduction proofs and the properties preserving proofs are deferred to the working paper [8].

7 Sponsored Search Syndicated Market: The Non-Separable Case for Sellers

7.1 S3M Characterization for Sellers

In this section we prove that the social choice function of any truthful S3M mechanism for sellers is an affine maximizer.

Recall that in the non-separable case the click-through rate can not be associated with a slot, i.e. seller, as it can not be presented as a simple product of the buyer click-through-rate and the slot click-through-rate. Different buyers might give different importance to different slots in the general case. Also recall that sellers in the S3M problem have a single private parameter which is their cost for the slot. Our sellers are a special case of a more general class of players:

DEFINITION 7.1. (Single Value Player) A single value player is a player that a single value determines his valuation. I.e., for a single value player j , good g_j and allocation a where $g_j \in a$, for all allocations e , $v_j(a) = v_j(e)$ if $g_j \in e$ and $v_j(e) = 0$ otherwise.

We first prove a generalization of the folk theorem that states that the design of a truthful mechanism for single-value-players requires critical price payment scheme and a monotone allocation, i.e., an allocation rule that if allocated a seller with cost $|c_j|$ it will also allocate the seller with any lower (absolute) cost $|\bar{c}_j| < |c_j|$.

LEMMA 7.1. Any individually rational mechanism for single value players is truthful if and only if its allocation rule is monotone and its payment scheme is based on critical price payment scheme.

Lemma 7.1's proof can be found in the working paper [8].

COROLLARY 7.1. Any monotonic S3M mechanism when applying critical price payment scheme to the sellers is a truthful mechanism for sellers.

Proof. Recall that the sellers in S3M are single value players, so the Corollary immediately follows from Lemma 7.1.

LEMMA 7.2. The social choice function of any truthful S3M mechanism is an affine maximizer for the sellers.

The proof of lemma 7.2 can be found in the working paper [8].

While lemma 7.2 shows that the social choice function of any truthful S3M mechanism is an affine maximizer for sellers, the model of sellers define in S3M might not be the natural one for slots with reserve prices in a sponsored search market. The S3M problem assumes that every slot is uniquely "owned" by a different seller, leading to an association between the click-through rate of a buyer and a seller, and not only to an association between the click-through rate of a buyer and a slot. A natural model of sellers in a sponsored search market is to only associate click-through rate of buyers with slots and to allow sellers to sell any slot. The specific slot sold by a seller is basically determined by his reported cost. Such model of sponsored search allocation is presented by [3, 18, 6]. While in sponsored search auctions where slots are zero this model maintains truthfulness, where slots are non-zero and no separability or extended separability⁴ condition are assumed the allocation monotonicity property does not maintain⁵ and therefore truthfulness will not hold for single-value sellers.

⁴for an explanation of extended separability see [12]

⁵a seller with a high (absolute) cost might be allocated before a seller with a low (absolute) cost as buyers click-through-rate might be much higher for the slot with the high cost than for the

A natural model that maintains monotonicity in the context of sponsored search market is a single seller selling slots in sponsored space. Such model was presented at [12] as a syndicated market where the selling side is a single syndicator such as LinkedIn selling sponsored search space to one of the search engines. An allocation to a single seller will maintain monotonicity as the higher (absolute) cost the seller expresses for his slots the less likely he is to sell them all. Unfortunately though the single seller model maintain monotonicity it is not a single-value-player, i.e., the seller has a different value for selling different number of slots out of all his slots for sale.

In order to extend the S3M model to a model with a single seller (syndicator) and to show that the extended model maintains truthfulness and affine maximization for the syndicator we perform a truth and non-affine maintaining reduction from the problem of the multi slot single seller problem (MSS) to the problem of multi sellers in S3M.

Problem 4. MSS: (*MSS - Multi-Slot Single Seller*)

Let $\check{G} = \{\check{g}_1, \dots, \check{g}_k\}$ be the set of goods and let \check{S} be a seller selling the goods at \check{G} . Let \check{S} have a vector of mappings $\check{C}(\check{c}_1, \dots, \check{c}_k)$ s.t. $\check{c}_j : A \rightarrow R^-$ such that $\check{c}_j(a) \leq 0$ if good \check{g}_j is allocated to a buyer and $\check{c}_{j+1}(a) \geq \check{c}_j(a)$ if good \check{g}_{j+1} is allocated to another buyer in allocation $a \in A$ and otherwise $\check{c}_j(a)=0$. The buyer side is defined the same as the buyer side in S3M. Let \check{M} be a set of pairs $(\check{b}_i, \check{g}_j)$, where $\check{b}_i \in \check{B}$ and $\check{g}_j \in \check{G}$, such that \check{M} is a matching between \check{B} and \check{G} .

We want to match buyers to goods such that the gain from trade of the allocation is maximized, i.e., our target function is

$$\max \sum_{i,j | (\check{b}_i, \check{g}_j) \in \check{M}} \check{\lambda}_{i,j} (\check{v}_i(a) + \check{c}_j(a)).$$

where a is the allocation immersed from \check{M} .

7.2 Reduction: MSS \leq S3M

In this subsection we proof that the MSS problem is reducible to the S3M problem.

Due to lack of space the technical details of the reduction construction, reduction proofs and the properties preserving proofs are deferred to the working paper [8].

slot with the low cost. Such buyers values of the click-through-rates can result in a non-monotone allocation rule where a non winning seller wins when he increases his (absolute cost)

8 The Non-Separable Case: Full Characterization (closing the open question at [12])

8.1 S3M Full Characterization

In this section we answer negatively the open question left at [12]; Does there exist a truthful budget balanced/surplus syndicated sponsored search market if separability or extended separability are not assumed? Or in other words is it possible to design a practical syndicated market i.e., a market where syndicators are selling their advertising space to the search engines without the search engines losing money in the market?

To answer the open question in [12] we relay on lemma in our working paper [8] that proves that the social choice function of any truthful S3M mechanism for buyers is an affine maximizer, and another lemma in our working paper [8] that proves that the social choice function of any truthful MSS mechanism for sellers is an affine maximizer. We make use of two lemmas introduced at [9] that integrate the truthful and non-affine properties maintained for buyer and sellers separately into truthful and non-affine properties maintained for all players. In addition we make use of the Myerson-Satterthwaite impossibility [16].

LEMMA 8.1. ([12]) *Given two disjoint sets of players B and S and an allocation a where each set has a price vector \bar{p}^B and \bar{p}^S supporting allocation a such that mechanism μ^B with allocation a is truthful for players in B under \bar{p}^B and mechanism μ^S with allocation a is truthful for players in S under \bar{p}^S , then the mechanism μ with allocation a is truthful for players in $B \cup S$ under the price vector which is the concatenation of the two price's vectors.*

LEMMA 8.2. ([12]) *Given two disjoint sets of players B and S and two affine maximizing social choice functions $\phi^B : V_B \Rightarrow A$, $\phi^S : V_S \Rightarrow A$, then the social choice function $\phi : V_B \times V_S \Rightarrow A$ is affine maximizing for players in $B \cup S$.*

THEOREM 8.1. *The social choice function of any truthful syndicated sponsored search market⁶ (MSS) mechanism is an affine maximizer.*

Proof. The proof immediately follows from the two lemmas in our working paper [8] (that prove that the social choice function of any truthful S3M mechanism for buyers is an affine maximizer, and that the social choice

⁶not assuming separability

function of any truthful MSS mechanism for sellers is an affine maximizer), together with Lemma 8.1, and Lemma 8.2. \square

COROLLARY 8.1. *There does not exist a truthful syndicated sponsored search market that is budget balanced/surplus and does not assume separability/extended separability property.*

Proof. Corollary 8.1 follows immediately from theorem 8.1 and the impossibility result of Myerson-Satterthwait [16]. Theorem 8.1 indicated that any truthful syndicated sponsored search market will maintain efficiency like property (affine maximization) while Myerson-Satterthwait impossibility states that not market can simultaneously maintain truthfulness, efficiency and budget balance properties. \square

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