Expressive Auctions for Externalities in Online Advertising*

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ABSTRACT

When online ads are shown together, they compete for user attention and conversions, imposing *negative externalities* on each other. We study the simplest form that an advertiser's valuation can have in the presence of such externalities– an advertiser's value depends on whether her ad is shown exclusively, or along with other ads. A mechanism such as the GSP auction, which always displays a full slate of ads, can be arbitrarily inefficient given such two-dimensional valuations; further, expanding the outcome space need not be enough– when the advertiser pays for an impression or a click, but derives value from a conversion, the private value of the advertiser is affected by the externality as well, leading to the need for a more expressive bidding language.

In this paper, we study the design of expressive mechanisms which are extensions of the GSP auction currently used in practice. Our mechanisms show either a single ad exclusively, or multiple ads simultaneously, with the property that the allocation and pricing are identical to GSP when multiple ads are shown. We investigate the equilibria of these mechanisms, and show that the revenue when multiple ads are shown dominates the VCG revenue. In equilibria where a single ad is shown, the revenue can actually be lower than that of VCG; however, this loss is bounded within a factor of two.

The increased efficiency from using the more expressive mechanism can come, unfortunately, with a loss in revenue with respect to the existing GSP mechanism in some cases. We design a mechanism which has a one-dimensional bidding language, while still allowing two types of outcomes, for situations where the *private* values are approximately one-dimensional. We show that this mechanism, while retaining the revenue and efficiency properties of the previous mechanisms, also revenue dominates the existing GSP mechanism.

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1. INTRODUCTION

Online advertisements shown alongside each other compete—first, for the user's attention, and then for a conversion. The effectiveness of an ad, therefore, depends not only on targeting it accurately to a relevant user, but also on the set of other advertisements that are displayed along with it: when high-quality competing ads are shown alongside an ad for a product, the chance that the user will eventually purchase the product is diminished. Online ads that are shown together thus impose negative externalities on each other.

This externality effect comes from two factors. First, the presence of other advertisements decreases the amount of attention an ad gets from a user: the user may not notice or click on an ad because of other competing ads. Second, even if a user notices or clicks on an ad, he may not convert on it, but instead convert on a competing advertisement (in fact, a user looking to purchase a product would arguably click on multiple ads before deciding on which one to convert on, and indeed, anecdotal evidence suggests that a significant fraction of ad clicks come from searches in which another ad was also clicked). Thus the presence of other ads affects not only the clickthrough rate, but potentially also the per click value of an ad. To account for such externalities, it is not enough to model and estimate the effect of other ads on clickthrough rates alone; rather, the outcomes and bidding languages offered by the auction mechanism must themselves be adequately expressive.

In this paper, we take the first steps towards designing mechanisms that are efficient in the presence of such externalities. In general, an advertiser's value in the presence of externalities is a function $\mathbf{v} : 2^n \rightarrow \mathrm{R}$.[1] We adopt the simplest version of such a valuation, where an advertiser's value depends on whether or not other advertisers are shown along with him, i.e., whether he is shown exclusively or not. Correspondingly, our mechanisms have two kinds of outcomes, one where a single ad is shown exclusively, and the other consisting of the standard slate of multiple ads displayed together.

Such a simplification is quite reasonable in the context of online advertising, for multiple reasons. First, expressing complex valuations imposes a heavy cognitive burden on advertisers (particularly the less sophisticated advertisers), who may not be able to, or want to, accurately determine their values for a wide range of outcomes. Second, the allocation problem when the value is a function of the identity of other advertisers becomes, in general, computa-

[1] n is the number of bidders.

tionally intractable and inapproximable, as can be seen by an easy reduction from independent set. Moreover, when the set of displayed ads are all chosen to be highly relevant to the same query or content, a reasonable first approximation is that the loss in value due to the externality effect depends mainly on the *number* of other ads displayed along with, rather than their identity. If k is the number of slots, this valuation would be represented by a (decreasing) k-dimensional vector— our two-dimensional valuation is a simple approximation, especially from an advertiser's point of view, for such a vector.

We next consider the question of a suitably expressive bidding language. In any model where an advertiser pays for an action (such as an impression or a click) which is different from the action she derives value from (a conversion), some part of the externality effect is incorporated into the private value of the advertiser, leading to the need for a two-dimensional bidding language. For instance, consider display advertising, which is sold on a pay per impression (CPM) basis. Since an advertiser ultimately derives value from conversions, and the rate of conversions conditional on an impression depends on whether or not other ads are shown along with, an advertiser's value per impression will be different depending on whether the ad is shown exclusively or with other advertisers. The same rationale applies to the pay per click models (CPC) used in sponsored search as well: advertisers pay per click, but derive value per conversion. If the probability of a conversion conditional on a click is different depending on whether or not other ads are simultaneously displayed, the advertiser's value per click will be different as well. Our mechanisms for display advertising (CPM) and sponsored search (CPC) therefore have a two-dimensional bidding language, where advertisers specify two bids, corresponding to their willingness to pay depending on whether they are shown exclusively or with other advertisers. However, in CPA models, or in CPC models where the change in value per click is negligible compared to the change in clickthrough rate in the presence of externalities, (that is, the change in the probability of conversion conditional on a click is negligible when multiple advertisers are shown together), a one-dimensional bidding language is adequate.

We will be interested in designing mechanisms that are *extensions* of the generalized second price auction (GSP) currently used to sell sponsored search ads, in the following sense: when the auction displays multiple ads together, we would like our mechanisms to use the same pricing and allocation rule as the GSP auction. Such auctions have the benefit that advertisers see no difference between the existing system and the new system when multiple ads are displayed together as before; they are also are easier to build and integrate with the existing system. Auctions which are extensions of GSP therefore have a better chance of actually being deployed in practice.

Results and Organization. We present auctions with expressive outcomes that are extensions of the GSP auction for CPM [2], CPC, and CPA-based systems in §3, §4 and §5, respectively. We analyze the efficiency and revenue properties of equilibria in each of these auctions. When multiple ads are displayed (in an envy-free equilibrium), the revenue from our auctions dominates that of VCG (note that this is *not* implied by the corresponding statement in [9] and [4], since the VCG payments in our model are different from (and larger than) the VCG payments in the model corresponding to the current GSP auction, due to the extra outcome possible). Unlike in [9] and [4], there can also exist equilibria with revenue lower than the VCG revenue; however, the loss in efficiency and revenue is bounded within a factor two in all "good" equilibria[3] (similar results hold for efficiency). For the CPM auction, we give a constructive proof that such good equilibria always exist in 3.1 and Theorem 3.1 (the result appears to be significantly harder to prove in the presence of clickthrough rates, and is work in progress.)

The increased efficiency from using the more expressive mechanism can come, unfortunately, with a loss in revenue with respect to the existing GSP mechanism, in situations where most advertisers's *private* values with and without exclusivity are similar . In §5, we design a mechanism for situations where the private values are approximately one-dimensional, which has a one-dimensional bidding language while still allowing two types of outcomes. We show that this mechanism, while retaining the revenue and efficiency properties of the previous mechanisms, also never suffers a loss in revenue with respect to the existing GSP mechanism.

Related Work. Externalities in online advertising have recently received plenty of attention in the literature, starting with the work in [5]. Externalities in sponsored search are studied by [1], [7], and [6], all based on a Markovian model of user clicking behavior: [1] and [7] study the algorithmic allocation problem in this model, while [6] analyzes the equilibria of the GSP mechanism with this model. These papers all model the externality effect via the effect on clickthrough rates, and do not address the issue of the conversion probability conditional on a click changing with the ads displayed. The work in [3] is more similar to ours in that it models the fact that conversion rates are conditional on clicks; however, it assumes a very specific form for the externality effect based on a postulated model of user behavior. Also, the paper focuses on analyzing equilibria for this model of conversion rates under the *existing* GSP mechanism, whereas we focus on designing more expressive mechanisms instead.

In [8], the authors study auctions with share-averse bidders, i.e., bidders suffering from negative externalities when item is shared amongst multiple competitors, exactly as in online advertising. However, while the authors study the problem of characterizing the revenue-maximizing single item auction for this setting, we are interested in an auction that exactly matches the current GSP auction when multiple slots are displayed.

A primary motivation for our work is the loss in efficiency due to limited expressiveness. The work in [2] provides a general theory for expressiveness in mechanisms, and relates the efficiency of mechanisms to their expressiveness in a domain independent manner.

2. MODEL

The model we use is the following. There are n advertisers bidding for a page containing k slots in which ads can be displayed. We use S to refer to an outcome where only a single ad is displayed on the page, and M to refer to an

[2]Display ads are increasingly being sold by auction, for instance, on the Right Media Exchange.

[3]Equilibria where losers bid at least their true value

outcome where multiple ads are displayed. (All k slots are filled when multiple ads are displayed, since advertisers do not express different values for filling fewer slots.) We denote the CTR of the i-th slot in outcome M by θ_i, and the CTR of the only slot in outcome S by $\overline{\theta}$; it is natural to expect $\overline{\theta} \geq \theta_1 \geq \ldots \geq \theta_k$. Our results also extend to the case of separable clickthrough rates, where the clickthrough rate of an ad in a slot is a product of an ad-dependent clickability and a slot-dependent clickability; we use the simpler model for clarity of exposition. Note that clickthrough rates are not relevant in §3, where advertisers bid and pay per impression.

Each advertiser has an underlying two-dimensional private valuation (v_i, v_i'): advertiser i has value v_i if he shown exclusively (corresponding to outcome S), and v_i' if other ads are shown as well (corresponding to outcome M). We assume that advertisers are ordered so that $v_1' \geq v_2' \geq \ldots \geq v_n'$. We will make the natural assumption that each advertiser (weakly) prefers exclusivity, i.e., $v_i \geq v_i'$. We will refer to the values v_i as S-values, and the values v_i' as M-values. (In §5, where $v_i = v_i'$ (i.e., when bidder valuations are essentially one-dimensional), we use v_i to denote the private values, and number advertisers so that $v_1 \geq \ldots \geq v_n$.)

Each advertiser's two-dimensional bid is denoted (b_i, b_i'), where b_i and b_i' represents her bids for outcomes S and M respectively. Again, we refer to the bids b_i as S-bids, and the bids b_i' as M-bids. We define indices $m_1, \ldots, m_n$ such that $b'_{m_1} \geq \ldots \geq b'_{m_n}$, i.e., according to the ordering of the M-bids. (In §5, where the bidding language is one-dimensional we use b_i to denote the bids, and the m_i are defined accordingly.) In the case of ties between advertisers, we will assume oracle access to the true values for tiebreaking[4].

We will use the indices max and max^2 to denote the bidders with the highest and second highest S-values, so $v_{\max} \geq v_i$ for every i, and $v_{\max^2} \geq v_i$ for every $i \neq \max$. We will also abuse notation to use $b_{\max}$ and $b_{\max^2}$ to denote the highest and second highest S-bids, respectively (in all equilibria of interest, these will actually correspond to the same bidders as with the true S-values).

A mechanism for this setting decides whether to allocate only one ad or multiple ads to be displayed on the page (i.e., S or M), and which advertisers to display. It then determines a payment for each winning advertiser. The VCG mechanism, of course, applies to each of these settings, and is a truthful mechanism which always produces an efficient (i.e., welfare maximizing) outcome. We will use VCG_{2D} to denote the VCG mechanism applied to the setting in §3 and §4, where bidder values are two-dimensional, and VCG_{1D} to denote VCG applied to the setting in §5, where bidder values are one-dimensional. Finally, we use VCG_M to denote VCG for the setting studied in [9], where the only possible outcome is M and advertisers have one-dimensional valuations. We make this distinction to easily distinguish between the VCG revenues in the various settings.

We will be interested only in equilibria where losers bid at least their true value, which we will refer to as "good" equilibria. This is particularly relevant when the outcome is S: while there might be Nash equilibria where the losing bidders must bid $b_i' < v_i'$ to ensure that the winner has no incentive to deviate to outcome M, it is unreasonable to expect that the losing bidders will not bid higher in an effort to change the outcome to M, which would give them positive utility. Thus Nash equilibria where the outcome is S but losers bid less than their true values simply to maintain equilibrium are unlikely to exist in practice.

[4]This assumption is made only for clarity of presentation and is not at all essential to the proofs.

3. CPM SYSTEMS

In this section, we begin with a setting where advertisers bid and pay per impression, i.e., on a CPM basis. That is, the values (v_i, v_i') and bids (b_i, b_i') are per impression values and bids respectively. This is the model most commonly used in display advertising, where brand advertisers are interested in simply displaying their advertisements to relevant users based on age, gender, demographics, etc. The importance of allowing expressive bidding is most evident in this setting, since the effectiveness of a display ad depends strongly on whether competing advertisements are shown along with or not.

The VCG mechanism for this setting is given below. Since it is a truthful mechanism, we use the values in place of bids when stating the allocation and payment rules.

EXAMPLE 3.1. *VCG_{2D}: The VCG mechanism first compares $v_{\max}$ and $v_1' + v_2' + \ldots + v_k'$.*

- *If $v_{\max} \geq v_1' + v_2' + \ldots + v_k'$, VCG_{2D} allocates the page to only one advertiser, namely* max*, and charges him the sum of the k highest v_i's (excluding himself) or the second highest* S*bid, whichever is larger. That is, if $1 \leq \max \leq k$, his payment is $\max(v_{\max^2}, v_1' + \ldots + v'_{\max-1} + v'_{\max+1} + \ldots + v'_{k+1})$, otherwise, he pays $\max(v_{\max^2}, v_1' + v_2' + \ldots + v_k')$.*
- *If $v_{\max} \leq v_1' + v_2' + \ldots + v_k'$, then the allocation will be* M*, but the pricing is more involved: it can be seen that the i-th advertiser (for $i \leq k$) has to pay $v'_{k+1} + \max(0, v_{\max_{-i}} - (V' - v_i'))$, where $\max_{-i}$ denotes the advertiser with highest* S *bid excluding advertiser i, and $V' = v_1' + \ldots + v'_{k+1}$. The expression roughly states that a winner should always pay at least v'_{k+1}, the highest rejected bid; in case the outcome changes to* S *with him dropping out, he should pay an extra amount equal to the difference in the welfare of* S *and* M *outcomes. In other words, the possibility of the* S *outcome can increase the price for winners in a* M *outcome as much as $\max(0, v_{\max_{-i}} - (V' - v_i'))$.*

Recall that we want to design a mechanism that is as similar as possible to GSP_M: in particular, we want the pricing and allocation to be exactly the same as GSP_M with bids b_i' if the outcome is M. We propose the following mechanism as a generalization of GSP for the new environment.

1. *GSP_{2D}: The mechanism GSP_{2D} compares $b_{\max}$ to $b'_{m_2} + b'_{m_3} + \ldots + b'_{m_{k+1}}$ to decide whether the outcome should be* S *or* M.

- *If $b_{\max} \geq b'_{m_2} + b'_{m_3} + \ldots + b'_{m_{k+1}}$, assign the page as a* S *to bidder* max*, and charge him $\max(b'_{m_2} + b'_{m_3} + \ldots + b'_{m_{k+1}}, b_{\max^2})$.*
- *If $b_{\max} \leq b'_{m_2} + b'_{m_3} + \ldots + b'_{m_{k+1}}$, assign the page to bidders $1, \ldots, k$ and charge them according to GSP_M.*

(We point out that in any equilibrium in which losing bidders bid at least their true value, the bidders $m_1, \ldots, m_k$ are exactly $1, \ldots, k$.)

Observe that the natural algorithm, which compares $b_{\max}$ with $b'_{m_1} + b'_{m_2} + \ldots + b'_{m_k}$, does not quite work: if the bidder 1, with the highest M-value, is different from the bidder max with the highest S-value, bidder 1 will always set $b'_1 = b_{\max} - \epsilon$ which changes the outcome to M *at no cost to bidder* 1 (as long as there is some other non-zero bid b'_i), since the pricing when the outcome is M according to GSP remains b'_2.

We next analyze the efficiency and revenue of equilibria in GSP_{2D}. The restriction to using GSP_M when the outcome is M does cause a loss in efficiency and revenue with respect to VCG_{2D}. However, as we show below, GSP_{2D} has fairly nice properties with respect to both efficiency and revenue.

We first prove that a *good* equilibrium always exists (recall that a good equilibrium is one where losers bid at least their true value).

Lemma 3.1. *Suppose the efficient outcome is* S*. There is an equilibrium with outcome* S *where losers bid at least their true values.*

Proof. Suppose the losers are bidding their true values. Note that the winner must be the bidder max with S-value $v_{\max}$. We show that max prefers outcome S to M if the efficient outcome is S.

$$v_{\max} \geq \sum_{i=1}^{k} v'_i \geq \sum_{i=2, i\neq \max}^{z} v'_i + v'_{\max} - v'_{k+1}$$

where z is chosen such that there are k terms in the summation. Rearranging, we get

$$v_{\max} - \sum_{i=2, i\neq \max}^{z} v'_i \geq v'_{\max} - v'_{k+1},$$

that is, max prefers S. The vector of bids $(\infty, 0)$ and (v_i, v'_i) for $i \neq \max$ is an equilibrium with outcome S. Note that if a loser bids ∞ to change the outcome to M, he has to pay at least v'_1 which leads to negative utility for him (recall that b'_1 does not affect the outcome). □

Our next result is technically more interesting, and constructs an equilibrium when the efficient outcome is M.

Theorem 3.1. *Suppose the efficient outcome is* M*. There is either an equilibrium with outcome* M*, or outcome* S *where losers bid at least their true values, or both.*

Proof Sketch. Suppose all bidders other than max bid truthfully. If max (weakly) prefers outcome S, then as in proof of 3.1, there exists vector of bids with outcome S where losers bid truthfully.

Now suppose max (strictly) prefers M if other bidders bid truthfully. According to GSP pricing, his payment for outcome M is at least v'_{k+1} and therefore, his utility is at most $v'_{\max} - v'_{k+1}$. Hence,

$$v_{\max} - \max\Big(\sum_{i=2, i\neq \max}^{z} v'_i, v_{\max^2}\Big) < v'_{\max} - v'_{k+1} \qquad (1)$$

where z is defined as in proof of lemma 3.1. Denote by $\bar{b}$ the vector of bids $(\bar{b}_i, \bar{b}'_i) = (v_i, v'_i)$ for $i \neq \max$, and $(\bar{b}_{\max}, \bar{b}'_{\max}) = (\infty, 0)$.

Now consider the vector of bids $\underline{b}$: $(\underline{b}_i, \underline{b}'_i) = (v_i, v'_{k+1})$ for $i \leq k+1$, and $(\underline{b}_i, \underline{b}'_i) = (v_i, v'_i)$ for all other bidders. If max prefers outcome M with bids $\underline{b}$, this is an equilibrium vector of bids (note that no other bidder prefers S, since his payment would be at least $v_{\max} \geq v_i$ for all i). If not, i.e. he strictly prefers outcome S with other bidders bidding according to $\underline{b}$:

$$v_{\max} - \max((k-1)v'_{k+1} + v'_{k+2}, v_{\max^2}) > v'_{\max} - v'_{k+1}. \qquad (2)$$

Note that the term v'_{k+2} appears in the price of max for S since max adjusts his M-bid to 0 if he is a winner with outcome S.

Note that the only difference between the inequalities (1) and (2) is the price of S, specifically, the term $\sum_{i=2, i\neq\max}^{z} b'_i$. We therefore increase the bids b'_i of the bidders $1, \ldots, k$ (excluding max) at the same time (and at the same rate) continuously, stopping for bidder i when $b'_i = v'_i$. Since $v_{\max} - \max(\sum_{i=2,i\neq\max}^{z} b'_i, v_{\max^2})$ is a continuous function of the b'_i's, from (2) and (1), there exists $\hat{b}$, $\underline{b} \leq \hat{b} \leq \bar{b}$ where equality holds, i.e.,

$$v_{\max} - \max\Big(\sum_{i=2, i\neq \max}^{z} \hat{b}'_i, v_{\max^2}\Big) = v'_{\max} - v'_{k+1} \qquad (3)$$

i.e., max is indifferent between S and M. Note that the vector of bids $\hat{b}$ has the form $(\hat{b}_i, \hat{b}'_i) = (v_i, \hat{v})$ for every $i \leq t$ excluding max (for some t and some $\hat{v}$), and $(\hat{b}_i, \hat{b}'_i) = (v_i, v'_i)$ for every $i > t$ again excluding max, and finally $(\hat{b}_{\max}, \hat{b}'_{\max}) = (\sum_{i=2}^{k+1} \hat{b}_i - \epsilon, v'_{k+1})$.

We prove that the vector of bids $\hat{b}$ is an equilibrium with outcome M. First, note that bidder max is indifferent between outcomes M and S, therefore, he has no incentive to deviate. Moreover, the outcome of this vector is M: in fact, $\hat{b}_{\max}$ is defined such that the outcome to be M. Furthermore, no other bidder can benefit from decreasing the bid because the outcome will switch to S leading to utility 0 for them. All we need to show is that no bidder other than max (for max we already know) can benefit from increasing b_i and winning the S outcome. Since inequality (2) is strict while we have equality in (3), we must have

$$\sum_{i=2, i\neq\max}^{z} \hat{b}'_i > v_{\max^2}.$$

Therefore, there is ϵ small enough such that

$$\hat{b}_{\max} = \sum_{i=2}^{k+1} \hat{b}_i - \epsilon > v_{\max^2}.$$

But $\hat{b}_{\max}$ is a lower-bound on the payment of any bidder for S, leading to negative utility for the bidder. □

Next, we investigate efficiency and revenue in good equilibria of GSP_{2D}. The proofs of each of these results follows from the proofs of the corresponding theorems in §4 and are omitted, since the pay per impression model is a special case of that in §4 with $\bar{\theta} = \theta_i = 1$.

Theorem 3.2 (Efficiency). *If the efficient outcome is* S *($v_{\max} > v'_1 + v'_2 + \ldots + v'_k$), there is no equilibrium of GSP_{2D} with outcome* M.

Together with Lemma 3.1, this result shows that when the efficient outcome is S, every good equilibrium of GSP_{2D} is efficient.

The same result does not hold when the efficient outcome is M; inefficient good equilibria with outcome S can indeed exist. However, the loss in efficiency in such equilibria is always bounded, as the following result shows.

PROPOSITION 3.1 (EFFICIENCY). *Suppose the efficient outcome is* M. *Every equilibrium of GSP_{2D} with outcome* S *where losers bid at least their true values has efficiency within a factor* 2 *of the optimal.*

Finally, we state the following result about the revenue of GSP_{2D} with respect to VCG_{2D}. Note that while GSP_M had the property that every envy-free equilibrium had revenue of at least as much as VCG_M, the same statement does not hold when comparing GSP_{2D} with VCG_{2D}, which is VCG applied to the two-dimensional valuations (the revenue from VCG_{2D} is greater equal that from VCG_M). The proof of the result below follows from the proofs of Theorems 4.2, 4.3 and 4.4 in §4 (observe that the loss in revenue actually occurs only when the outcome of GSP_{2D} is S; when the outcome is M, the revenue of GSP_{2D} is at least as much as the revenue of (the two-dimensional) VCG_{2D}).

THEOREM 3.3 (REVENUE). *Any envy-free equilibrium of GSP_{2D} with outcome* M *has revenue at least as much as VCG_{2D}. Any equilibrium of GSP_{2D} with outcome* S *where losers bid at least their true value has revenue at least half that of VCG_{2D}.*

4. CPC SYSTEMS

We now move on to CPC-based systems, where advertisers bid and pay per click, as is prevalent in sponsored search. The values (v_i, v'_i), and bids (b_i, b'_i) are correspondingly per-click values and per-click bids respectively.

Note that when advertisers have such two-dimensional valuations, a mechanism such as GSP_M which always displays multiple ads (i.e., has only one outcome, M) can suffer from an unbounded loss in efficiency (for instance, if $v'_i = \epsilon v_i$ for all advertisers, i.e., all advertisers strongly prefer exclusivity).

In the presence of CTRs, GSP_{2D} is modified as follows.

GSP_{2D} with clickthrough rates: The mechanism GSP_{2D} compares $b_{\max}\overline{\theta}$ to $\sum_{i=2}^{k+1} \theta_{i-1} b'_{m_i}$ to decide whether the outcome should be S or M.

- If $b_{\max}\overline{\theta} \geq \sum_{i=2}^{k+1} \theta_{i-1} b'_{m_i}$, the outcome is S with winning bidder max, whose payment is $\frac{\sum_{i=2}^{k+1} \theta_{i-1} b'_{m_i}}{\overline{\theta}}$ per click.
- If $b_{\max}\overline{\theta} \leq \sum_{i=2}^{k+1} \theta_{i-1} b'_{m_i}$, assign the page to bidders $m_1, \ldots, m_k$ and charge them according to GSP pricing, i.e. bidder m_i (for $i \leq k$) has to pay $b'_{m_{i+1}}$ per click.

In the remainder of this section, we investigate the efficiency and revenue of the equilibria of this mechanism. As before, we will be interested in these properties for good equilibria of the mechanism, where losers bid at least their true value. Further, when the outcome is M, we will restrict ourselves, as in [9] and [4], to *envy free* equilibria, since the revenue guarantees for GSP_M relative to VCG_{1D} themselves hold only for envy-free equilibria of GSP_M.

The easy lemma below, which follows immediately from individual rationality, will be used repeatedly.

LEMMA 4.1. *In any equilibrium of GSP_{2D} with outcome* M, *$v'_{m_i} \geq b'_{m_{i+1}}$ for every $i \leq k$.*

THEOREM 4.1. *If the efficient outcome is* S *($v_{\max}\overline{\theta} > \sum_{i=1}^{k} \theta_i v'_i$), there is no equilibrium of GSP_{2D} with outcome* M.

PROOF. Assume for the sake of contradiction that b' is an equilibrium vector of bids in GSP_{2D} with outcome M and recall that indices $m_1, \ldots, m_n$ are such that $b'_{m_1} \geq \ldots \geq b'_{m_n}$.

First, note that by Lemma 4.1, $v_{\max}\overline{\theta} > \sum_{i=1}^{k} \theta_i v'_i \geq \sum_{i=1}^{k} \theta_i b'_{m_{i+1}}$. Therefore, if the bidder with highest S value, max, bids his true value, the outcome will change to S. Consequently, max must be in $\{m_1, \ldots, m_k\}$, since otherwise, he has utility 0 in the outcome M while he can make his utility positive by bidding truthfully.

Next, we show that even if max $\in \{m_1, \ldots, m_k\}$, he still has an incentive to deviate and bid truthfully to change the outcome to S. More precisely, we show that the payoff of max in outcome S is more his payoff in outcome M:

$$\overline{\theta}(v_{\max} - \frac{(\theta_1 b'_{m_2} + \ldots + \theta_k b'_{m_{k+1}})}{\overline{\theta}}) > \theta_j(v'_{\max} - b'_{m_{j+1}}),$$

where index j is such that $m_j = \max$. That is, we want to show

$$\overline{\theta} v_{m_j} - (\theta_1 b'_{m_2} + \ldots + \theta_k b'_{m_{k+1}}) > \theta_j(v'_{m_j} - b'_{m_{j+1}}).$$

Rearranging, and using the fact that max $\in \{m_1, \ldots, m_k\}$, that is, $j \leq k$, it suffices to prove:

$$\overline{\theta} v_{m_j} > (\theta_1 b'_{m_2} + \ldots + \theta_{j-1} b'_{m_j} + \theta_j v'_{m_j} + \theta_{j+1} b'_{m_{j+2}} + \ldots + \theta_k b'_{m_{k+1}}).$$

To show this, we start with:

$$v_{\max} > \frac{\theta_1 v'_1 + \ldots + \theta_k v'_k}{\overline{\theta}}.$$

By definition $v'_1 \geq \ldots \geq v'_n$, therefore we get:

$$v_{\max} > \frac{\theta_1 v'_{m_1} + \ldots + \theta_k v'_{m_k}}{\overline{\theta}}.$$

By Lemma 4.1, we can replace v'_{m_i} by $b'_{m_{i+1}}$ for all $i \neq j$, which gives us

$$v_{m_j} > \frac{(\theta_1 b'_{m_2} + \ldots + \theta_{j-1} b'_{m_j} + \theta_j v'_{m_j} + \theta_{j+1} b'_{m_{j+2}} + \ldots + \theta_k b'_{m_{k+1}})}{\overline{\theta}},$$

which is what we needed to show. □

PROPOSITION 4.1. *Suppose the efficient outcome is* M. *Every equilibrium of GSP_{2D} with outcome* S *where losers bid at least their true values has efficiency within a factor* 3 *of the optimal. Any envy-free equilibrium with outcome* M *is efficient.*

PROOF. For brevity and clarity, we describe the proof when bidder max is bidder 1 (the proof for j is very similar). Since the outcome of GGSP is S, she must prefer outcome S to every slot; specifically, to the first slot in outcome M. Therefore,

$$\overline{\theta} v_{\max} - \sum_{i=1}^{k} \theta_i b'_{m_{i+1}} \geq \theta_1(v'_1 - b'_{m_2}).$$

Since losers bid at least their true values $m_i \leq i+1$ and therefore, $b'_{m_i} \geq v'_{i+1}$, so we get

$$\overline{\theta} v_{\max} \geq \theta_1 v'_1 + \sum_{i=2}^{k} \theta_i v'_{i+2}.$$

Our goal is to show $3\overline{\theta} v_{\max} \geq \sum_{i=1}^{k} \theta_i v'_i$ which follows from the above inequality because $\overline{\theta} v_{\max} \geq \theta_1 v'_1$, $\overline{\theta} v_{\max} \geq \theta_2 v'_2$, and $\theta_1 v'_1 + \sum_{i=2}^{k} \theta_i v'_{i+2} \geq \sum_{i=3}^{k} \theta_i v'_i$. The efficiency of an envy-free equilibrium of GSP_{2D} follows directly from the arguments in [9]. □

While an argument similar to that in the proof of Theorem 5.2 can be used to show the existence of an equilibrium (although without the guarantee on revenue), note that losers need not be bidding at least their true value in these equilibria. While we were able to prove the existence of good equilibria in the previous section without clickthrough rates, this appears to be harder when clickthrough rates are present. Therefore, unlike in the previous section, where we guaranteed the existence of equilibria with good revenue and efficiency properties, here we only guarantee that such equilibria have good revenue and efficiency properties *if* they exist. We point out that when $\overline{\theta}$ is either sufficiently high or sufficiently low (i.e., close to θ_1), it is easy to show that such good equilibria do exist; however, the meaning of 'sufficiently' depends not only on the other clickthrough rates, but on the underlying true values as well.

4.1 Revenue

In this section, we compare the revenues of equilibria in GSP_{2D} with the revenue of VCG_{2D}. We point out that the results in this section make use of the assumption that advertisers weakly prefer exclusivity, i.e., $v_i \geq v'_i$. Before we move on to the revenue comparison results, we restate the allocation and pricing rule of the VCG mechanism for this setting with clickthrough rates. The expressions for the VCG prices, and correspondingly the proofs of our results, are considerably less neat than in the previous section with no clickthrough rates.

EXAMPLE 4.1. ***VCG with CTR's*** *The VCG mechanism compares* $\overline{\theta} v_{\max}$ *and* $\theta_1 v'_1 + \theta_2 v'_2 + \ldots + \theta_k v'_k$.

- *If* $\overline{\theta} v_{\max} \geq \theta_1 v'_1 + \theta_2 v'_2 + \ldots + \theta_k v'_k$, *VCG allocates the page to only one advertiser, namely* max, *and charges him either the sum of the* k *highest* $\theta_i v'_i$*s (excluding himself) divided by* $\overline{\theta}$ *or the second highest* S *value, whichever is larger. That is, if* $1 \leq \max \leq k$ *the winner's payment is* $\max(v_{\max^2}, \frac{\theta_1 v'_1 + \ldots + \theta_{\max -1} v'_{\max -1} + \theta_{\max +1} v'_{\max +1} + \ldots + \theta_{k+1} v'_{k+1}}{\overline{\theta}})$ *per click, else it is* $\max(v_{\max^2}, \frac{\theta_1 v'_1 + \theta_2 v'_2 + \ldots + \theta_k v'_k}{\overline{\theta}})$ *per click.*

- *If* $\overline{\theta} v_{\max} < \theta_1 v'_1 + \theta_2 v'_2 + \ldots + \theta_k v'_k$, *then VCG allocation is* M, *but the expression for the payments is more complicated. The i-th advertiser's payment per click is (for* $i \leq k$*):*

$$p_i = \max(\sum_{j=i}^{k} (\theta_j - \theta_{j+1}) v'_{j+1}, \overline{\theta} v_{\max_{-i}} - \sum_{j=1}^{k} \theta_j v'_j + \theta_i v_i)$$

THEOREM 4.2. *Suppose the efficient outcome is* S. *The revenue in any equilibrium of* GSP_{2D} *where losers bid at least their true values is at least half of the revenue of* VCG_{2D}.

PROOF. First, recall that the only possible equilibrium outcome is S. The revenue of GSP_{2D} is $\max(\overline{\theta} b_{\max^2}, \sum_{i=1}^{k} \theta_i b'_{m_{i+1}})$. We give lower-bounds for both terms and then show that the revenue of VCG_{2D} cannot be larger than the sum of the lower bounds; therefore, the revenue of VCG_{2D} cannot be more than twice of the revenue of GSP_{2D}.

First we assume $\max \neq 1$. We have $\overline{\theta} b_{\max^2} \geq \overline{\theta} v_{\max^2} \geq \overline{\theta} v_1 \geq \overline{\theta} v'_1 \geq \theta_1 v'_1$, since $v'_i \leq v_i$ and losers bid at least their true values.

For the other term, we know that all bidders except max are losers in outcome S. Therefore, $b'_i \geq v'_i$ for every $i \neq \max$. So we get

$$\sum_{i=1}^{k} \theta_i b'_{m_{i+1}} \geq \sum_{i=2}^{j-1} \theta_{i-1} v'_i + \sum_{i=j+1}^{k+2} \theta_{i-2} v'_i$$

where $j = \min(\max, k+2)$.

On the other hand, the revenue of VCG_{2D} is

$$\max(\overline{\theta} v_{\max^2}, \sum_{i=1}^{l-1} \theta_i v'_i + \sum_{i=l+1}^{k+1} \theta_{i-1} v'_i)$$

where $l = \min(\max, k+1)$.

To finish the proof, we need to show that the sum of the lower bounds we have for GSP_{2D} is greater than or equal to revenue of VCG_{2D}:

$$\overline{\theta} v_{\max^2} + \sum_{i=2}^{j-1} \theta_{i-1} v'_i + \sum_{i=j+1}^{k+2} \theta_{i-2} v'_i \geq$$
$$\max(\overline{\theta} v_{\max^2}, \sum_{i=1}^{l-1} \theta_i v'_i + \sum_{i=l+1}^{k+1} \theta_{i-1} v'_i).$$

If the first term in the VCG_{2D} revenue is the dominant term, the inequality obviously holds. Otherwise, we need to show that

$$\overline{\theta} v_{\max^2} + \sum_{i=2}^{j-1} \theta_{i-1} v'_i + \sum_{i=j+1}^{k+2} \theta_{i-2} v'_i \geq \sum_{i=1}^{l-1} \theta_i v'_i + \sum_{i=l+1}^{k+1} \theta_{i-1} v'_i,$$

i.e., it is enough to show that

$$\theta_1 v'_1 + \sum_{i=2}^{j-1} \theta_{i-1} v'_i + \sum_{i=j+1}^{k+2} \theta_{i-2} v'_i \geq \theta_1 v'_1 + \sum_{i=2}^{l-1} \theta_i v'_i + \sum_{i=l+1}^{k+1} \theta_{i-1} v'_i,$$

but this inequality clearly holds using term-by-term comparison.

It remains to prove the the theorem for the case where $\max = 1$. In this case, the revenue of GSP_{2D} is actually greater equal that of VCG_{2D}. The revenue of GSP_{2D} is

$$\max(\overline{\theta} b_{\max^2}, \sum_{i=1}^{k} \theta_i b'_{m_{i+1}}) \geq \max(\overline{\theta} b_{\max^2}, \sum_{i=1}^{k} \theta_i v'_{i+1})$$

while the revenue of VCG_{2D} is

$$\max(\overline{\theta} v_{\max^2}, \sum_{i=1}^{k} \theta_i v'_{i+1}).$$

Therefore, the revenue of VCG_{2D} is less than or equal to the revenue of GSP_{2D} in this case.

□

A simple modification to Example 4.2 shows that this factor of 2 is tight (set $v_1 = 3$ so that the efficient outcome is S).

The following additive bound on revenue follows immediately from the previous proof:

$$R_{VCG_{2D}} - \theta_1 v_1' + \theta_k v_{k+2}' \leq R_{GSP_{2D}}.$$

THEOREM 4.3. *Suppose the efficient outcome is* M. *Any envy-free equilibrium of* GSP_{2D} *with outcome* M *has revenue greater equal that of* VCG_{2D}.

PROOF. First, note that since the equilibrium is envy-free, the ordering of M-bids is the same as ordering of M-values ([9]). We show that the payment of advertiser i (for $i \leq k$) in GSP_{2D} is at least as much as his payment in VCG_{2D}.

Recall from 4.1 that the payment for advertiser i in VCG_{2D} is

$$p_i = \max(\sum_{j=i}^{k}(\theta_j - \theta_{j+1})v_{j+1}', \overline{\theta} v_{\max_{-i}} - \sum_{j \neq i}^{k} \theta_j v_j').$$

First we prove that GGSP payment of bidder i, $\theta_i b_{i+1}'$, is at least $\overline{\theta} v_{\max_{-i}} - \sum_{j \neq i}^{k} \theta_j v_j'$. We will prove this by contradiction: if not, we show that there is a bidder with a profitable deviation to S. Let l be the bidder with the highest S-value excluding i, i.e. $v_l = v_{\max_{-i}}$. By the contradiction hypothesis,

$$\theta_i b_{i+1}' < \overline{\theta} v_l - \sum_{j \neq i}^{k} \theta_j v_j'.$$

If l is not a winner, he has a profitable deviation by bidding (v_l, b_l') which changes the outcome to S because $\overline{\theta} v_l > \sum_{j \neq i}^{k} \theta_j v_j' + \theta_i b_{i+1}' \geq \sum_{j=1}^{k} \theta_j b_{j+1}'$. (Of course, bidding $(v_l, 0)$ is a "more profitable" deviation, but is unnecessary for the argument.)

So suppose that l is a winner. Adding and subtracting $\theta_l b_{l+1}'$ and rearranging we get

$$\theta_l(v_l' - b_{l+1}') < \overline{\theta} v_l - (\theta_i b_{i+1}' + \theta_l b_{l+1}' + \sum_{j \neq i,l}^{k} \theta_j v_j').$$

Note that the term in parentheses on the right hand side is an upper-bound on the price that l has to pay for S if he deviates and bids (v_l, b_l'): the price for S is at most $\sum_{j=1}^{k} \theta_j b_{j+1}'$ (since the outcome with the original vector of bids was M, $\overline{\theta} b_{\max} \leq \sum_{j=1}^{k} \theta_j b_{j+1}'$, so the price for S is always dominated by this term). Since $b_{j+1}' \leq v_j'$ (the original vector of bids was in equilibrium), the price for S is upper-bounded by $(\theta_i b_{i+1}' + \theta_l b_{l+1}' + \sum_{j \neq i,l}^{k} \theta_j v_j')$ as claimed, showing that l can deviate profitably. (Note that as before, this bid does change the outcome to S.)

The fact that $\theta_i b_{i+1}' \geq \sum_{j=i}^{k}(\theta_j - \theta_{j+1})v_{j+1}'$ follows from the lower bound on bids in envy-free equilibria in [9], which also holds for envy-free equilibria in outcome M of GSP_{2D}. □

THEOREM 4.4. *Suppose the efficient outcome is* M. *The revenue in any equilibrium of* GSP_{2D} *with outcome* S *where losers bid at least their true values is at least half of the revenue of* VCG_{2D}.

PROOF. The proof, unfortunately, proceeds by considering cases. The revenue of GSP_{2D} is $\max(\overline{\theta} b_{\max^2}, \sum_{i=1}^{k} \theta_i b_{m_{i+1}}')$. The revenue of VCG_{2D} is $\sum_{i=1}^{k} p_i$ where

$$p_i = \max(\sum_{j=i}^{k}(\theta_j - \theta_{j+1})v_{j+1}', \overline{\theta} v_{\max_{-i}} - \sum_{j \neq i}^{k} \theta_j v_j').$$

For ease of notation let $p_i^1 \sum_{j=i}^{k}(\theta_j - \theta_{j+1})v_{j+1}'$, and $p_i^2 = \overline{\theta} v_{\max_{-i}} - \sum_{j \neq i}^{k} \theta_j v_j')$, so that $p_i = \max(p_i^1, p_i^2)$. First note that $p_i^1 \leq \theta_i v_{i+1}'$. Also, from individual rationality we have $p_i \leq \theta_i v_i'$.

We consider the following three cases, and will prove for each case that $\overline{\theta} b_{\max^2} + \sum_{i=1}^{k} \theta_i b_{m_{i+1}}' \geq \sum_{i=1}^{k} p_i$. Therefore, the revenue of GSP_{2D} is at least half the revenue of VCG_{2D}.

1. If $\max > k$:

   All bids $b_1', \ldots, b_{k+1}'$ are at least v_i' in this case, so the revenue of GSP_{2D} is at least $\max(\overline{\theta} v_{\max^2}, \sum_{i=1}^{k} \theta_i v_{i+1}')$. The revenue of VCG_{2D} is at most $p_1 + \sum_{i=2}^{k} \theta_i v_i'$. Since $\overline{\theta} v_{\max^2} \geq \overline{\theta} v_1 \geq \overline{\theta} v_1' \geq \theta_1 v_1'$, we have $\overline{\theta} v_{\max^2} \geq p_1$, and hence

   $$\overline{\theta} v_{\max^2} + \sum_{i=1}^{k} \theta_i v_{i+1}' \geq p_1 + \sum_{i=2}^{k} \theta_i v_i',$$

   which implies that the revenue of GSP_{2D} is at least half the revenue of VCG_{2D}.

2. If $1 < \max \leq k$: The revenue of GSP_{2D} in this case is at least $\max(\overline{\theta} v_{\max^2}, \sum_{j=1}^{\max -2} \theta_j v_{j+1}' + \sum_{j=\max -1}^{k} \theta_j v_{j+2}')$ because all losers bid at least their true values. We first consider the case where $p_i^1 \geq p_i^2$ for every i. The revenue of VCG_{2D} cannot be more than $\sum_{j=1}^{k} p_j^1 \leq \sum_{j=1}^{k} \theta_j v_{j+1}'$. Since $\overline{\theta} v_{\max^2} \geq \overline{\theta} v_1' \geq \overline{\theta} v_{\max}' \geq \theta_{\max} v_{\max}'$,

   $$\overline{\theta} v_{\max^2} + \sum_{j=1}^{\max -2} \theta_j v_{j+1}' + \sum_{j=\max -1}^{k} \theta_j v_{j+2}' \geq \sum_{j=1}^{k} \theta_j v_{j+1}',$$

   which shows the revenue of VCG_{2D} cannot be more than twice the revenue of GSP_{2D} in this case.

   For the other case, let l be some index for which $p_l^1 < p_l^2$. We consider two cases depending on whether $p_{\max}^1 > p_{\max}^2$ or $p_{\max}^2 \geq p_{\max}^1$. For both cases, we upper-bound the VCG_{2D} payment of bidder i (for $i \neq \max$ and $i \neq l$) by $\theta_i v_i'$. First, if $p_{\max}^2 \geq p_{\max}^1$, the revenue of VCG_{2D} is at most

   $$\begin{aligned} p_l^2 + p_{\max}^2 + \sum_{j \neq \max, j \neq l}^{k} \theta_j v_j' &= \overline{\theta} v_{\max} - \sum_{j \neq l}^{k} \theta_j v_j' \\ &+ \overline{\theta} v_{\max^2} - \sum_{j \neq \max}^{k} \theta_j v_j' + \sum_{j \neq \max, j \neq l}^{k} \theta_j v_j' \\ &= \left(\overline{\theta} v_{\max} - \sum_{j=1}^{k} \theta_j v_j'\right) + \overline{\theta} v_{\max^2}. \end{aligned}$$

   Since the efficient outcome is M, the term in parentheses is less equal zero, therefore, the revenue of VCG_{2D}

is bounded above by $\overline{\theta}v_{\max^2}$, which is clearly less equal the revenue of GSP_{2D}.

Now, if $p^1_{\max} \geq p^2_{\max}$, the revenue of VCG_{2D} is

$$p^2_l + p^1_{\max} + \sum_{j\neq \max, j\neq l}^{k} \theta_j v'_j$$

$$= \overline{\theta}v_{\max} - \sum_{j\neq l}^{k} \theta_j v'_j + \sum_{j=\max}^{k} (\theta_j - \theta_{j+1})v'_{j+1} + \sum_{j\neq \max, j\neq l}^{k} \theta_j v'_j.$$

Since $\overline{\theta}v_{\max} - \sum_{j\neq l}^{k} \theta_j v'_j \leq \theta_l v'_l$, the revenue of VCG_{2D} is at most

$$\sum_{j=1}^{\max -1} \theta_j v'_j + \sum_{j=\max}^{k} \theta_j v'_{j+1} = \theta_1 v'_1 + \sum_{j=2}^{\max -1} \theta_j v'_j + \sum_{j=\max}^{k} \theta_j v'_{j+1}.$$

Since $\overline{\theta}v_{\max^2} \geq \theta_1 v'_1$, by term-by-term comparison we get

$$\overline{\theta}v_{\max^2} + \sum_{j=1}^{\max -2} \theta_j v'_{j+1} + \sum_{j=\max -1}^{k} \theta_j v'_{j+2} \geq \theta_1 v'_1 + \sum_{j=2}^{\max -1} \theta_j v'_j + \sum_{j=\max}^{k} \theta_j v'_{j+1},$$

which implies the revenue of GSP_{2D} is at least half of the revenue of VCG_{2D}.

3. If $\max = 1$:

The revenue of GSP_{2D} in this case is at least $\max(\overline{\theta}v_{\max^2}, \sum_{j=1}^{k} \theta_j v'_{j+2})$ because all losers bid at least their true values. As before, we first consider the case where $p^1_i \geq p^2_i$ for every i; the revenue of VCG_{2D} cannot be more than $\sum_{j=1}^{k} p^1_j \leq \sum_{j=1}^{k} \theta_j v'_{j+1}$. Since $\overline{\theta}v_{\max^2} \geq \overline{\theta}v_2 \geq \overline{\theta}v'_2 \geq \theta_1 v'_2$,

$$\overline{\theta}v_{\max^2} + \sum_{j=1}^{k} \theta_j v'_{j+2} \geq \theta_1 v'_2 + \sum_{j=2}^{k} \theta_j v'_{j+1}$$

which shows that the revenue of VCG_{2D} cannot be more than twice of revenue of GSP_{2D} in this case.

The analysis of the other case is almost identical to when $1 < \max \leq k$, so we omit repeating it here.

□

Example 4.2 shows that this factor 2 is tight as well.

How does GSP_{2D} compare to GSP_M in terms of revenue? Suppose bidders have two-dimensional valuations (v_i, v'_i), but are only offered the GSP_M mechanism with its one-dimensional bidding language. Since the outcome of GSP_M is never S, bidders will bid according to valuations v'_i in GSP_M. The example below shows that the revenue of GSP_{2D} can actually be smaller than the revenue in GSP_M, i.e., if the search engine had persisted with the old mechanism. Note that the example continues to hold for $v_1 = M$ for any $M > 1$; demonstrating a revenue efficiency trade-off: choosing to use GSP_M leads to greater revenue than GSP_{2D}, but can lead to an arbitrary loss in efficiency. However, in this example, the losing bidders do not particularly value exclusivity, that is, their preferences are approximately one-dimensional. In the next section, we design a mechanism with two outcomes, but a one-dimensional bidding language, for valuations where bidders do not place a high premium on exclusivity. As we will see, the new mechanism does not suffer from this potential loss in revenue with respect to GSP_M.

EXAMPLE 4.2. *Suppose that $\overline{\theta} = 1 + \epsilon$ and $\theta_i = 1$ for $i = 1, 2$, $k = 2$, $v_1 = 1+\epsilon, v_2 = v_3 = 1$, and $v'_i = 1$ for $i \leq 3$. The revenue of GSP_M for this example is 2 for all equilibria and all bidders have utility 0. However, if advertiser 1 bids $(\infty, 0)$, and advertisers 2 and 3 bid truthfully, this is an equilibrium with revenue 1 and payment 1 with utility $\epsilon > 0$ for the winner. In fact, this is the highest possible revenue in any equilibrium outcome of GSP_{2D}.*

5. CPA SYSTEMS

As we saw in the previous section, a two-dimensional bidding language can cause loss in revenue relative to GSP_M when most bidders do not value exclusivity highly. In this section, we study an auction with a one-dimensional bidding language but two outcomes S and M as before, for situations where $v_i \approx v'_i$ for all bidders, since there is little loss in efficiency here in allowing bidders to report only one value. Such an auction is preferable when bidders' private values are one-dimensional, as in cost per conversion systems, or approximately one-dimensional, as in CPC models where the externality imposed by other ads affects mainly the click-through rate and has a negligible effect on the value per click of an ad. Note that the auction is still expressive–it has two possible outcomes S and M, and in fact an advertiser's total value is still two-dimensional; the bid is one-dimensional because it reflects the advertiser's private value (as distinct from the total value), which is one-dimensional.

Recall that bidders have one dimensional values v_i, and are numbered so that $v_1 \geq \ldots \geq v_n$. Also, in this section we will assume that $\overline{\theta} > \theta_1$, since otherwise there is no benefit from allowing the outcome S.

The mechanism GSP_{1D} takes as input a vector of bids $(b_1, \ldots, b_n)$, and computes the allocation and payments as follows.

- If $\overline{\theta}b_{m_1} \geq \sum_{i=1}^{k} \theta_i b_{m_{i+1}}$ then the outcome is S. The price that advertiser 1 has to pay is $\max(\overline{\theta}b_{m_2}, \sum_{i=1}^{k} \theta_i b_{m_{i+1}})$.
- Otherwise, the outcome is M and the pricing is the same as in GSP_M.

5.1 Efficiency

THEOREM 5.1. *If the efficient outcome is S ($\overline{\theta}v_1 > \sum_{i=1}^{k} \theta_i v_i$), there is no envy-free equilibrium of GSP_{1D} with outcome M. Further, there exist equilibria with outcome S.*

PROOF. Suppose instead that there is an envy-free equilibrium $b_1, \ldots b_{k+1}$ where the outcome is M. Since the equilibrium is envy-free, from [9], the first advertiser must occupy the top slot. Since he prefers not to deviate to S we have

$$\overline{\theta}v_1 - \sum_{i=1}^{k} \theta_i b_{i+1} \leq \theta_1(v_1 - b_2).$$

Also, since the bids are in equilibrium, we have $b_{i+1} \leq v_i$ from individual rationality. Therefore,

$$\overline{\theta} v_1 \leq \sum_{i=1}^{k} \theta_i v_i$$

which contradicts the efficiency condition.

To show the existence of an equilibrium with outcome S, let $b_1, \ldots, b_{k+1}$ be an (envy-free) equilibrium vector of bids in GSP_M(existence is shown in [9]). We will show that the first bidder prefers the outcome S with any such vector of bids, and therefore, he can bid ∞ to ensure this outcome (none of the losers has incentive to deviate because they cannot change the outcome to increase their utility).

Since $b_{i+1} \leq v_i$, we have $\sum_{i=1}^{k} \theta_i v_i \geq \theta_1 v_1 + \sum_{i=2}^{k} \theta_i b_{i+1}$. Since the efficient outcome is S, $\overline{\theta} v_1 \geq \sum_{i=1}^{k} \theta_i v_i \geq \theta_1 v_1 + \sum_{i=2}^{k} \theta_i b_{i+1}$. Rearranging,

$$\overline{\theta} v_1 - \sum_{i=1}^{k} \theta_i b_{i+1} \geq \theta_1 (v_1 - b_2) \geq \theta_i (v_1 - b_{i+1})$$

where the last inequality is because the bids $b_1, \ldots, b_{k+1}$ is an envy-free equilibrium in GSP_M. So the first bidder prefers S to any slot in M, and the vector $(\infty, b_2, \ldots, b_{k+1})$ is an equilibrium for GSP_{1D}. □

EXAMPLE 5.1. *If the efficient outcome is* M*,* GSP_{1D} *can have equilibria which are inefficient, as this example shows. Suppose there are three slots* $k = 3$ *with* $\theta_1 = 1, \theta_2 = 1, \theta_3 = 1/2$ *and four bidders with* $v_1 = 2, v_2 = 3/2, v_3 = 3/2, v_4 = 1$*; suppose that* $\overline{\theta} = 2 + \epsilon$*.*

The efficient outcome is M *with welfare* 4.25 *while the welfare of* S *is* $4 + 2\epsilon$*.*

We show that the vector of bids $(\infty, 3/2, 3/2, 1)$ *is an equilibrium for* GSP_{1D} *with outcome* S*. First, note that the utility of advertiser* 1 *for* S *outcome is* $4 + 2\epsilon - 3.5 > 0.5$*. However, his utility for each of the slots in outcome* M *is exactly* 0.5 *which implies he prefers* S *outcome to* M*. No other advertiser can change the outcome from* S *to* M *because* $\overline{\theta} > \theta_1$*.*

Note that Theorem 5.2 guarantees the existence of an equilibrium (with outcome either S or M) when the efficient outcome is M. We prove the result below for efficiency of "good equilibria", i.e., equilibria when losers bid at least their true values.

PROPOSITION 5.1. *Suppose the efficient outcome is* M*, every equilibrium of* GSP_{1D} *with outcome* S *where losers bid at least their true values has efficiency within a factor* 2 *of the optimal. Any envy-free equilibrium with outcome* M *is efficient.*

PROOF. From [9], every envy-free equilibrium with outcome M is efficient also. This only leaves us with equilibria where the outcome is S, but the efficient outcome is M.

Since S is the equilibrium outcome of GSP_M, advertiser 1 must prefer outcome S to M. In particular,

$$\overline{\theta} v_1 - \sum_{i=1}^{k} \theta_i v_{i+1} \geq \theta_1 (v_1 - v_2).$$

Rearranging and adding $\overline{\theta} v_1$ to both sides, we have

$$\overline{\theta} v_1 + \overline{\theta} v_1 \geq \sum_{i=2}^{k} \theta_i v_{i+1} + \theta_1 v_1 + \overline{\theta} v_1,$$

and therefore

$$2\overline{\theta} v_1 \geq \sum_{i=1}^{k} \theta_i v_i,$$

i.e., the welfare in outcome S is never less than half that in the efficient outcome. □

5.2 Revenue

Recall that in Example 4.2, the highest revenue equilibrium of GSP_{2D} had strictly smaller revenue than the revenue in GSP_M. The theorem below shows that this cannot happen with GSP_{1D}: for every equilibrium of GSP_M, there is an equilibrium of GSP_{1D} with at least as much revenue.

THEOREM 5.2. *Suppose* $b_1, \ldots, b_{k+1}$ *is an envy-free equilibrium vector of bids for* GSP_M*, there is an equilibrium vector of bids for* GSP_{1D} *with at least the same revenue.*

PROOF. Suppose that the underlying vector of values is $v_1, \ldots, v_{k+1}$.

Given the bids $b_2, \ldots, b_{k+1}$, advertiser 1 prefers either S or M. We consider each of these cases. First suppose he (strictly) prefers M. Since $b_1, \ldots, b_{k+1}$ is an envy-free equilibrium of GSP_M, if he (strictly) prefers M he must have the highest utility for the first slot. So

$$\overline{\theta} v_1 - \sum_{i=1}^{k} \theta_i b_{i+1} < \theta_1 (v_1 - b_2)$$

We will show that $\tilde{b_1} = b_2 + \epsilon, b_2, b_3, \ldots, b_{k+1}$ is an envy-free equilibrium vector of bids in GSP_{1D} leading to outcome M. Rearranging the condition above we get

$$\sum_{i=2}^{k} \theta_i b_{i+1} > v_1 (\overline{\theta} - \theta_1) \geq b_2 (\overline{\theta} - \theta_1)$$

so

$$\overline{\theta} b_2 < \sum_{i=1}^{k} \theta_i b_{i+1}.$$

So there is small enough ϵ such that $\overline{\theta} \tilde{b_1} < \sum_{i=1}^{k} \theta_i b_{i+1}$ and the outcome is M. Furthermore, this is an envy-free equilibrium because changing b_1 does not affect the price of any slot and the other bids are unchanged. Also, the revenue in GSP_M with this vector of bids is the same as that in GSP_M with bids $b_1, \ldots, b_{k+1}$.

Next, suppose he prefers S. Then by bidding high enough, (*i.e.*, the bid vector is $(\infty, b_2, \ldots, b_{k+1})$) he can change the outcome to S. Note that since $\overline{\theta} > \theta_1$, no S deviation by any of the bidders $2, \ldots, k+1$ can change the outcome to M, so these bidders have no incentive to deviate. So this is an equilibrium with revenue $\max(\overline{\theta} b_2, \sum_{i=1}^{k} \theta_i b_{i+1})$ which is at least as large as the revenue in GSP_M.

□

Note that the condition for advertiser 1 to prefer M given $b_2, \ldots, b_{k+1}$ can be written as

$$\sum_{i=1}^{k} \theta_i b_{i+1} > (\overline{\theta} - \theta_1) v_1 + \theta_1 b_2.$$

So if there exists envy-free equilibria in GSP_M with high enough revenue, or equivalently, if $\overline{\theta}$ is 'small', *i.e.*, close enough to θ_1, there is an envy-free equilibrium in GSP_{1D} with outcome M. Note that when $\overline{\theta}$ is 'high' enough, the advertiser with the highest value will prefer outcome S even when losers bid truthfully. Thus for small enough, and large enough $\overline{\theta}$, "good" equilibria exist; a constructive proof is required only for intermediate values of $\overline{\theta}$.

Next we compare the revenue of GSP_{1D} with that of VCG applied to this setting.

PROPOSITION 5.2. *In any equilibrium of GSP_{1D} with outcome* S *where losers bid at least their true values, the revenue of GSP_{1D} is at least half the revenue of VCG_{1D}.*

PROOF. The payment of advertiser i (for $i \leq k$) in VCG is $p_i = \max(\sum_{j=i}^{k}(\theta_j - \theta_{j+1})v_{j+1}, \overline{\theta}v_{\max_{-i}} - \sum_{j \neq i}^{k} \theta_j v_j)$. On the other hand, revenue of GSP_M is $\max(\sum_{i=1}^{k} \theta_i v_{i+1}, \overline{\theta}v_2)$.

By individual rationality we have $p_i \leq \theta_i v_i$. Moreover, for advertiser 1, $\sum_{j=i}^{k}(\theta_j - \theta_{j+1})v_{j+1} \leq \theta_1 v_2$, and $v_{\max_{-i}} - v_2$, so $p_1 \leq \max(\theta_1 v_2, \overline{\theta}v_2) \leq \overline{\theta}v_2$ (recall that $\overline{\theta} > \theta_1$). Therefore, the revenue of VCG_{1D} is at most $\overline{\theta}v_2 + \sum_{i=2}^{k} \theta_i v_i$. Now

$$2R_{GSP_{1D}} \geq \overline{\theta}v_2 + \sum_{i=1}^{k} \theta_i v_{i+1} \geq \overline{\theta}v_2 + \sum_{i=2}^{k} \theta_i v_i = R_{VCG_{1D}},$$

which completes the proof. □

THEOREM 5.3. *Suppose the efficient outcome is* M. *Any envy-free equilibrium of GSP_{1D} with outcome* M *has at least as much revenue as VCG_{1D}.*

The proof of this statement is very similar to that of Theorem 4.3, and is omitted for want of space.

6. FURTHER WORK

The most obvious question left unanswered in our current work is the question of the existence of good equilibrium in pay per click models (recall that we prove the existence of such good equilibria for pay per impression models, where click through rates are not present). There are a number of other interesting directions for further work as well. The form of values we consider, where an advertiser has one value for being shown exclusively and one for being shown with other advertisers, is only one way to approximate a k- dimensional value vector which is a decreasing function of the number of accompanying ads. Another two-dimensional approximation to the same underlying function is that an advertiser has some value v_i as long as no more than some number n_i of other ads are shown alongwith, and zero if any more than n_i ads are shown. Which of these is a better bidding language? The answer to this must depend on the form of the underlying k-dimensional value vector, and formalizing the relation could be an interesting theoretical question.

There are interesting empirical questions as well. How much do accompanying ads affect the conversion rate conditional on a click, *i.e.*, how much does the value per click change with or without accompanying ads? What do the clickthrough rates $\overline{\theta}$ and θ_i actually look like, and what are the revenue and efficiency properties in equilibria with real parameter values? The answers to these questions will be important to determine which of these mechanisms are suitable for use in practice.

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